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HW 6

Math 583B
April 16, 2008

1. Show that the set of λ for which $|z| = 1$ is the interval $[-4, 0]$.

   Solution. Look at the eigenvalue problem

$$q_{m+1} - 2q_m + q_{m-1} = \lambda q_m \tag{1}$$

   and make the subsitution $q_m = e^{im\theta}$. The above equation becomes

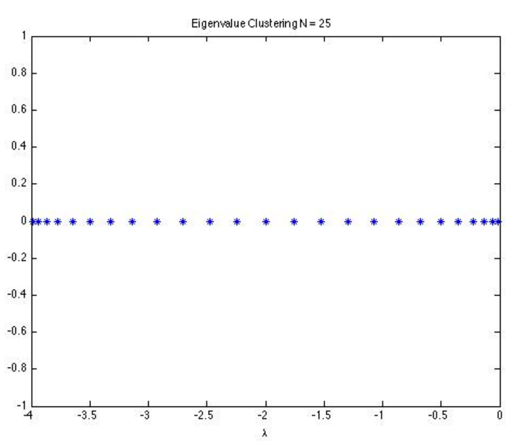
$$\begin{aligned} \lambda e^{im\theta} &= e^{m\theta}(e^{i\theta} - 2 + e^{-i\theta}) & (2) \\ \lambda &= (e^{i\theta} + e^{i\theta} - 2) & (3) \\ &= 2\cos\theta - 2 & (4) \end{aligned}$$

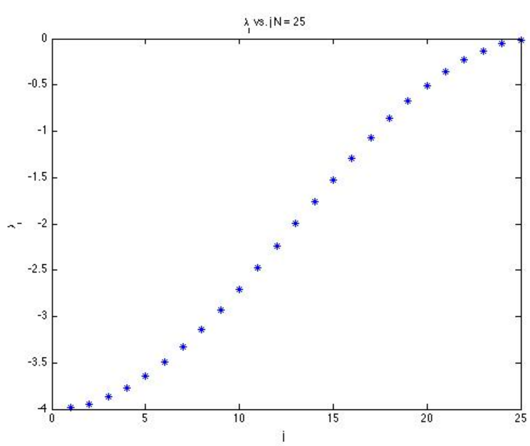
   Since the *min* and *max* of the *cos* function are -1 and 1, respectively, $\lambda \in [-4, 0]$.

2. Consider finite problem

   a.) Plot the eigenvalues in two ways: 1.) as points along the horizontal axis and 2.) plot $\lambda_j vs. j$.

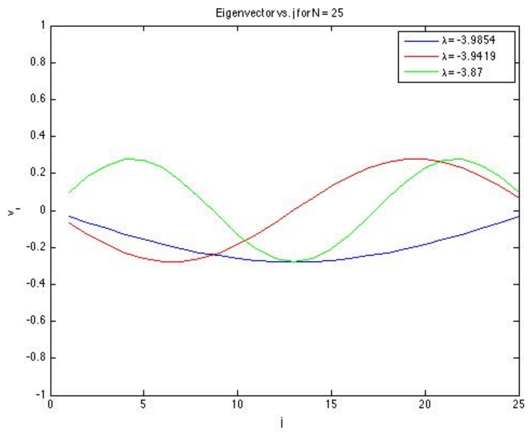
   Solution. The two plots are:

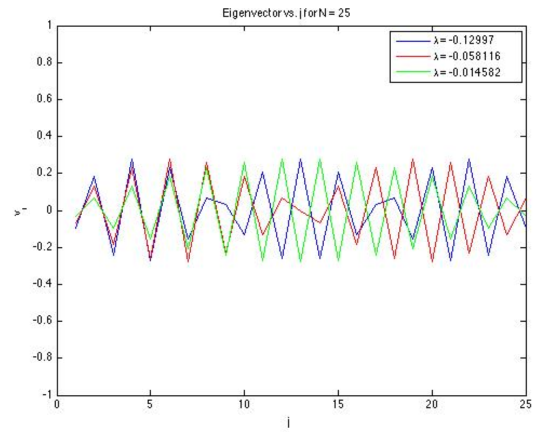




   You can see the clustering from the slope of the λ_j curve vs. j. As the slope flattens out, the eigenvalues become more dense.

   b.) Plot eigenvectors and show the oscillation/interlacing property.



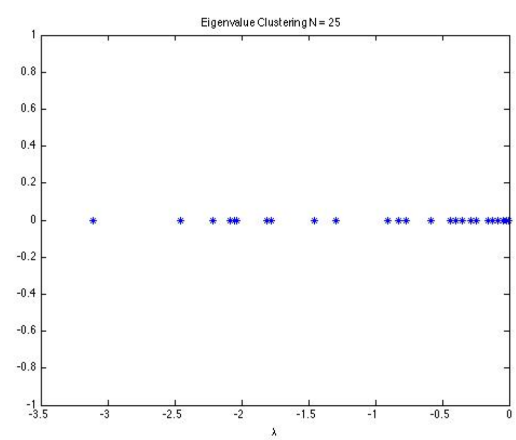


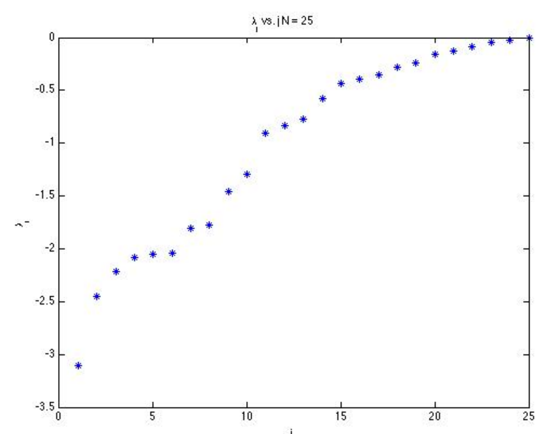
Solution. The two plots are above.

It is clear from the first plot that the eigenvector corresponding to the j^{th} eigenvalue has $(j-1)$ zeros. That is, each successive eigenvector has one more zero than the previous. From the second plot, the zeros can be seen to overlap each other, that is the j^{th} and the $(j+1)^{th}$ eigenvector zeros interlace.

3. Repeat the previous problem for random positive k_j. Verify the oscillation theorem. a.)

   Solution. The two plots are:

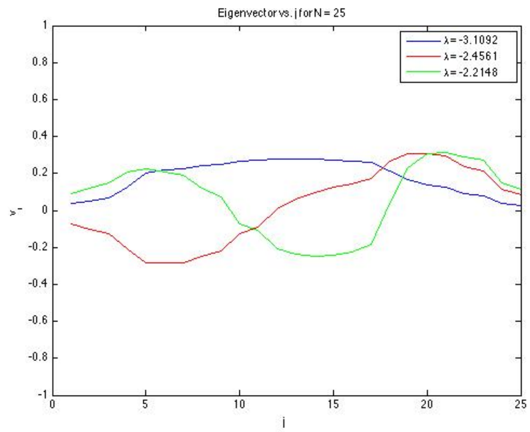


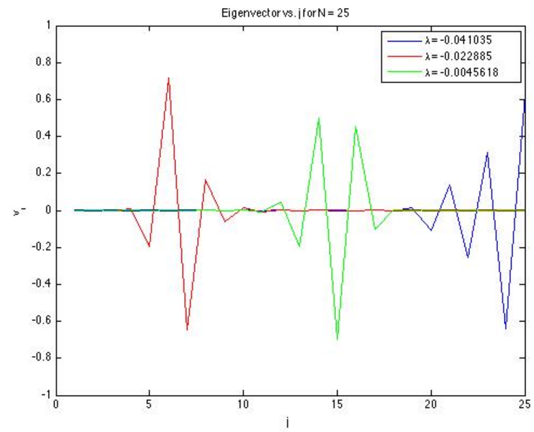


As before you can see the clustering from the slope of the λ_j curve vs. j. As the slope flattens out, the eigenvalues become more dense.

b.)

Solution. The two plots are below.

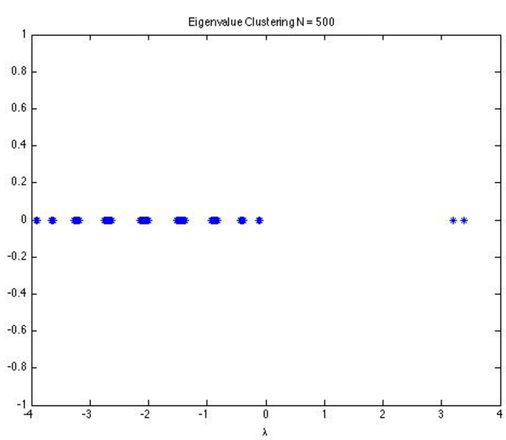


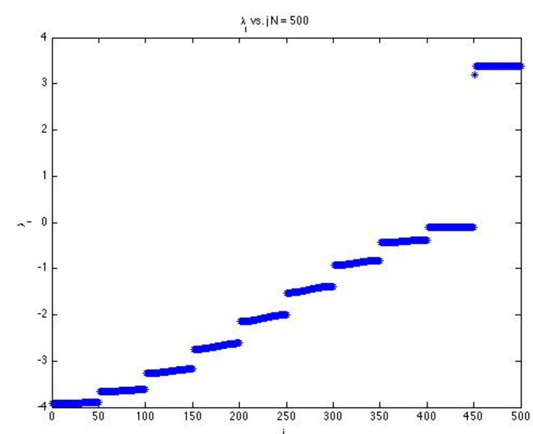


Despite the randomness of the matrix entries, the first plot shows that the eigenvector corresponding to the j^{th} eigenvalue has $(j-1)$ zeros. That is, each successive eigenvector has one more zero than the previous. From the second plot, the zeros can be seen to overlap each other, that is the j^{th} and the $(j+1)^{th}$ eigenvector zeros interlace.

4. Take a 500 x 500 matrix A with $k_j = 1$, and add periodic diagonal matrix V where every 10^{th} entry is 5. Let H=A+V. Repeat the previous exercise with the matrix H. a.)

   Solution. The two plots are:

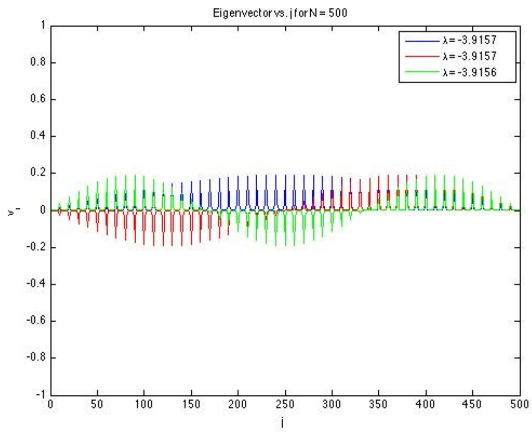


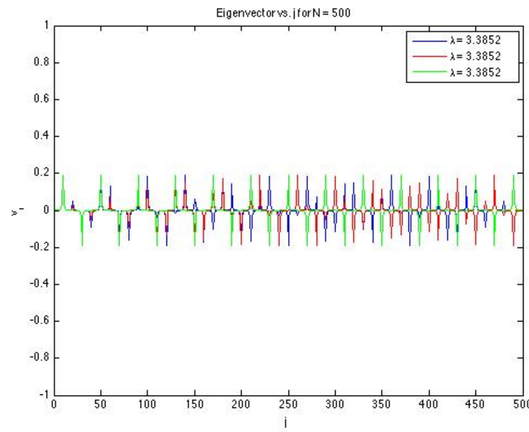


It is difficult to see the clustering of the eigenvalues from the second plot, however the greater the slope of the discrete, discontinuous lines λ_j vs. j the wider the eigenvalue spread on the x-axis.

b.)

Solution. The two plots are below.

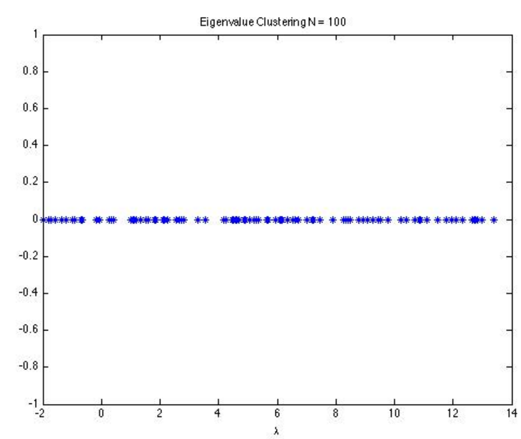


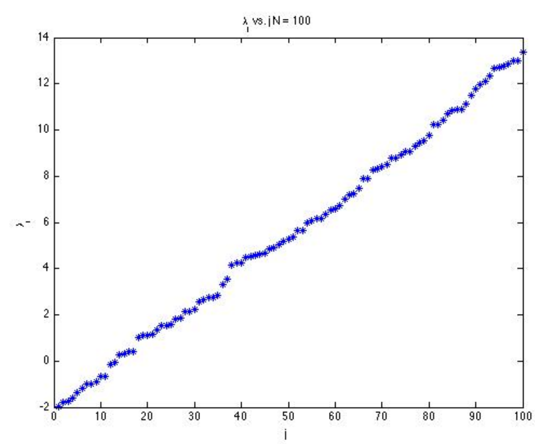


The eigenfunctions look similar to those of number 2, except now the eigenvectors are sharply influenced by the periodic potential. However, the oscillation theorem still holds. The zeros, although difficult to see, overlap (look at the right chart for $j < 200$ or so).

5. Take A to be 100 x 100 matrix with $k_j = 1$, and add diagonal V whose entries are random. Plot the eigenvalue spectrum in two ways and several interesting eigenvectors.

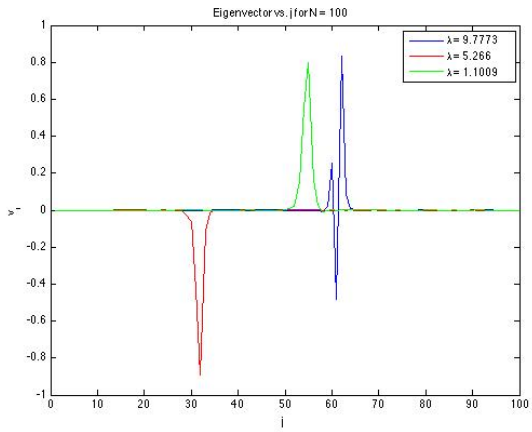
Solution. The two plots are:





The eigenvalues look to be distributed evenly across the entire spectrum domain. The slope of the plot above on the right is close to $\frac{max\lambda_j}{j}$. The plot above on the left shows the same thing.

A few interesting eigenvectors are:



These eigenvectors, as expected, do not obey the oscillation theorem.